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THE PROBLEM OF THE BOUNDEDNESS OF THE CONTACT PRESSURES ON THE OUTLINE OF AN ELLIPTICAL STAMP INTERACTING WITH AN ELASTIC LAYER*

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The possibility of constructing a bounded solution of the integral equation of the contact problem for an elastic layer in the case of an elliptical stamp with a polynomial base is investigated. It is shown that the contact domain for a parabolic stamp turns out to be elliptical if the relative thickness of the layer is fairly large or fairly small.

It is well-known that the solution of the contact problem of the theory of elasticity without taking account of the friction force with a fixed previously assigned contact domain, results in the general case in infinite values of the contact pressures on the contact domain contour. The hypothesis of the boundedness of the contact pressure distribution function on the contact domain contour can be used as the necessary condition that the solution of the problem must satisfy.

Since a function of fairly general form can be approximated to an arbitrary degree of accuracy by a certain polynomial, the question of whether it is possible to construct a solution, bounded in a certain contact domain, for the integral equation of the contact problem for an elastic layer with a polynomial free term will be completely natural.

The question of whether such a solution exists is investigated in this paper in the case of an elliptical contact area.

1. Formulation of the problem. Without taking friction between the stamp and the layer, and the layer and the base, into account, the fundamental integral equation of the contact problem for an elastic layer can be written in the form /1/

$$\int\int q(\xi,\eta)\,K(R/h)\,d\xi d\eta = 2\pi h\theta f(x,y),\ (x,y)\in\Omega \tag{1.1}$$

$$K(t)=\int_0^\infty \frac{\operatorname{ch}2u-1}{\operatorname{sh}2u+2u}J_0(ut)\,du,\quad R=\sqrt{(\xi-x)^2+(y-\eta)^2},$$

$$\theta=G(1-\nu)^{-1}$$

$$f(x,y)=\delta+\alpha x+\beta y-g(x,y)>0,\ (x,y)\in\Omega$$

Here and henceforth, the double integrals are taken over the contact domain Ω between the stamp and the layer, $q(x,y)$ is the contact pressure, h is the layer thickness, $J_0(x)$ is the Bessel function, G and ν are elastic constants of the layer, $\delta+\alpha x+\beta y$ is the rigid displacement of the stamp under the action of applied force P, and $g(x,y)$ is a function describing the shape of the stamp base.

The equilibrium condition for the stamp

$$P=\iint q(\xi,\eta)\,d\xi d\eta,\ Pe_1=\iint q(\xi,\eta)\,\xi d\xi d\eta,\ Pe_2=\iint q(\xi,\eta)\,\eta d\xi d\eta \tag{1.2}$$

must be appended to the integral equation (1.1).

Here e_1 and e_2 are projections of the eccentricity of the application of the force P on the x and y axes.

The solution of (1.1) yielding a minimum of the functional /2/

$$I=\iint q(\xi,\eta)\,f(\xi,\eta)\,d\xi d\eta \tag{1.3}$$

for a given form of the function $f(x,y)$ and small possible variations of the domain Ω must be found in problems with a variable contact domain. The condition mentioned is necessary and sufficient to determine the boundary L of the contact domain Ω; is particular, /3/ we obtain from it as a necessary condition the boundedness of the contact pressure $q(x,y)$ on L. In the case when the function $f(x,y)$ satisfies the Hölder condition on L, this condition takes the simple form

$$q(x,y)=0,\ (x,y)\in L \tag{1.4}$$

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We note that after the contact problem has been solved, the obvious physical conditions for the solution must be verified: the displacements of points of the layer surface outside the contact domain Ω are such that it never intersects the surface of the stamp; $q(x, y) \geqslant 0$ for $(x, y) \in \Omega$.

We will rewrite the integral equation (1.1) as follows /1/:

$$\iint q(\xi, \eta) \frac{d\xi\, d\eta}{R} = 2\pi\theta f(x, y) + \frac{1}{h} \iint q(\xi, \eta) F\left(\frac{R}{h}\right) d\xi\, d\eta, \quad (x, y) \in \Omega \tag{1.5}$$
$$F(t) = t^{-1} - K(t), \; 0 \leqslant t < \infty$$

where the function $F(t)$, continuous with all its derivatives, can be expanded in an absolutely convergent series for $0 \leqslant t < 2$

$$F(t) = \sum_{i=0}^{\infty} a_i t^{2i}, \quad a_i = \frac{(-1)^i}{(2i!)^2} \int_0^{\infty} \left[1 - \frac{\operatorname{ch} 2u - 1}{\operatorname{sh} 2u + 2u}\right] u^{2i}\, du \tag{1.6}$$

Values of the constants a_i are given in Table 3 in /1/.

We now consider the integral equation

$$A[q(\xi, \eta)] = 2\pi\theta f(x, y), \quad A[\ldots] = \iint (\ldots) \frac{d\xi\, d\eta}{R}, \quad (x, y) \in \Omega \tag{1.7}$$

obtained from (1.5) for $\lambda = h/a \to \infty$ $(a = {}^1/_2 \max_\Omega R)$ and the corresponding contact problem for an elastic half-space. From experience in solving plane and axisymmetric contact problems for an elastic half-space /1/, Theorems 23.2 and 40.2), it can be asserted that in the general spatial case of (1.7), the correctness relationship

$$\| q(x, y) \|_{L(\Omega)} \leqslant 4\pi\theta a m_\Omega \| f(x, y) \|_{M^2(\Omega)} \tag{1.8}$$

holds when the contour L of the odmain Ω is fairly smooth.

Here $L(\Omega)$ is the space of functions absolutely summable in Ω, and $M^2(\Omega)$ is the space of functions whose second derivatives are bounded in Ω. We define the norm in $M^2(\Omega)$ by the relationship

$$\| f \|_{M^2(\Omega)} = \max |f| + a \max [|f_x'| + |f_y'|] + a^2 \sup [|f_x''| + |f_{xy}''| + |f_y''|] \tag{1.9}$$

Inequality (1.8) means that an inverse operator A^{-1}, bounded from $M^2(\Omega)$ in $L(\Omega)$ exists for an integral operator A of the form (1.7), where $\| A^{-1} \| \leqslant 2am_\Omega$. For a set of functions $q(x, y) \in L(\Omega)$ such that $q(x, y) \geqslant 0$ in Ω, the constant is $m_\Omega < 1$. This results from (1.7).

We now consider the integral equation (1.5). We will show that if its solution exists in $L(\Omega)$, then it can be obtained by successive approximations for $\lambda > \lambda_*$. To do this, we act on both sides of (1.5) with the operator A^{-1}. We obtain

$$q = 2\pi\theta A^{-1}[f] + B[q], \quad B[q] = A^{-1}\left[\frac{1}{h} \iint q(\xi, \eta) F\left(\frac{R}{h}\right) d\xi\, d\eta\right] \tag{1.10}$$

We will prove that the operator B in the space $L(\Omega)$ is a compression operator under the condition $\lambda > \lambda_*$. Then the above-mentioned assertion will result from the Banach principle. We have the estimate

$$\| B[q^{(1)}] - B[q^{(2)}] \|_{L(\Omega)} \leqslant \frac{2}{\lambda} m_\Omega \Big[\max |F(t)| + \frac{\sqrt{2}}{\lambda} \max |F'(t)| +$$
$$\frac{2}{\lambda^2} \max |F''(t)| + \frac{\sqrt{3}}{\lambda^2} \max |F'(t)| t^{-1}\Big] \iint |q^{(1)} - q^{(2)}|\, d\xi\, d\eta =$$
$$k \| q^{(1)} - q^{(2)} \|_{L(\Omega)}$$

We now find λ_* from the equation $k = 1$ as the least positive root. We note that, by (1.5) and (1.6),

$$\max |F(t)| = a_0, \; \max |F'(t)| < 0.5819 \cdot 2\, |d_1|, \tag{1.11}$$
$$\max |F'(t)| t^{-1} = \max |F''(t)| = 2\, |a_1|$$

Here $0.5819 = \max J_1(x)$, where $J_1(x)$ is the Bessel function and the constants are $a_0 = 1.168$; $d_1 = -0.521$; $a_1 = -0.395$ (/1/, Table 3). Taking account of (1.11) and the inequality $m_\Omega < 1$, which should hold if the problem is posed correctly physically (i.e., compliance with the condition $q(x, y) \geqslant 0$ is assured in Ω; note that this is known to be so if the shape of the stamp base is convex), we obtain $k < k_*$, $k_* = 5.897\lambda^{-3} + 1.715\lambda^{-2} + 2.336\lambda^{-1} = 1$; hence we find $\lambda_* = 3.37$.

2. Solution for a large relative thickness of the layer. We limit ourselves to the first M terms in the expansion of $F(t)$ in the form (1.6). Then substituting this segment of the series instead of $F(t)$ into (1.5), we will have

$$\iint q(\xi, \eta) \frac{d\xi\, d\eta}{R} = 2\pi\theta f(x, y) + \sum_{i=0}^{M} \frac{a_i}{h^{2i+1}} \iint q(\xi, \eta) R^{2i}\, d\xi\, d\eta, \quad (x, y) \in \Omega \tag{2.1}$$

We will construct a solution of integral equation (2.1) bounded in the elliptic domain Ω (semi-axes a and b) if the shape of the stamp base is given by a polynomial of degree N, i.e.

$$g(x, y) = \sum_{k,\, l=0}^{k+l \leqslant N} b_{kl} x^k y^l \tag{2.2}$$

Note that a polynomial of degree

$$t = \max(N, 2M) \tag{2.3}$$

is on the right side of (2.1).

Then, as is known /1/, the solution of (2.1) bounded in L for the elliptical domain Ω should, if it exists, have the form

$$q(x, y) = \sum_{m,\, n=0}^{m+n \leqslant t-2} a_{mn} x^m y^n \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2} \tag{2.4}$$

Substituting (2.4) into (2.1) and integrating on the right and left sides /1/, we obtain a relationship interconnecting two polynomials of degree t whose coefficients depend on a_{mn} and b_{kl}. Equating coefficients of identical powers of x and y, we obtain a system of $^1/_2(t + 1)(t + 2)$ independent algebraic equations. These equations contain $^1/_2(t - 1)t$ unknown coefficients a_{mn}. Of the $^1/_2(N + 1)(N + 2)$ coefficients b_{kl}, $^1/_2(N - 1)N$ should be arbitrary, while the remaining $2N + 1$ are additional unknowns for the given quantities δ, α, β, a and b. The requirement that the $^1/_2(N - 1)N$ coefficients b_{kl} should be arbutrary is necessary so that the solution of (1.7) can be obtained from the solution of (2.1) in the special case as $\lambda \to \infty$.

Therefore, the system of $^1/_2(t + 1)(t + 2)$ equations contains $^1/_2(t - 1)t + 2N + 1$ unknowns. For $N \geqslant 2M$ it follows from (2.3) that $t = N$, and therefore, the number of unknowns agrees with the number of equations in the system. For $2M > N$ we will have $t = 2M$, and the unknowns in the system are less than the number of equations. It can hence be concluded that if the base of the line is a polynomial of degree $N = 2p$, then the solution of (1.5) that is bounded on the contour of the elliptical contact domain, can be obtained for large λ only to terms $O(\lambda^{-2p-3})$.

We shall now seek the solution of (1.5) in the form /1/

$$q(x, y) = \sum_{n=0}^{\infty} q_n(x, y) h^{-n} \tag{2.5}$$

Substituting (2.5) into both sides of the integral equation (1.5) and equating terms in identical powers of h, we obtain an infinite system of integral equations of the form (1.7) for the sequential determination of the functions $q_n(x, y)$

$$\begin{aligned}
&\iint q_0(\xi, \eta) \frac{d\xi\, d\eta}{R} = 2\pi\theta f(x, y), \ (x, y) \in \Omega \\
&\iint q_1(\xi, \eta) \frac{d\xi\, d\eta}{R} = a_0 \iint q_0(\xi, \eta)\, d\xi\, d\eta \\
&\iint q_2(\xi, \eta) \frac{d\xi\, d\eta}{R} = a_0 \iint q_1(\xi, \eta)\, d\xi\, d\eta \\
&\iint q_3(\xi, \eta) \frac{d\xi\, d\eta}{R} = \iint [a_0 q_2(\xi, \eta) + a_1 q_0(\xi, \eta) R^2]\, d\xi\, d\eta \\
&\iint q_4(\xi, \eta) \frac{d\xi\, d\eta}{R} = \iint [a_0 q_3(\xi, \eta) + a_1 q_1(\xi, \eta) R^2]\, d\xi\, d\eta, \ldots
\end{aligned} \tag{2.6}$$

Since the series (1.6) converges for $t < 2$, or for $\lambda > 1$, the expansion of the solution in the form (2.5) can be constructed for $\lambda > \sup(\lambda_*, 1)$. However, we note that the estimated value obtained in Sect.1 for λ_* is exaggerated. If we limit ourselves to keeping terms of the order $O(\lambda^{-4})$ in (2.5), then the results can be used for $\lambda \geqslant 1.52$ (/1/, Table 36) and the error will not exceed 5%.

Now, as above, let the contact domain be an ellipse with semiaxes a and b in planform, and let the shape of the stamp base be described by the function $g(x, y) = Ax^2 + By^2, A > 0, B > 0$ (parabolic stamp). Let the impressing force P be applied to the stamp centre of symmetry, i.e., $e_1 = e_2 = 0$, then also $\alpha = \beta = 0$. Solving the integral equations (2.6) sequentially, we construct /1/ the asymptotic solution on the contour of the elliptical domain Ω to terms $O(\lambda^{-5})$ for the case mentioned:

$$q(x, y) = \frac{\theta}{b}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2} \left\{D \left[1 + \frac{s}{h} + \left(\frac{s}{h}\right)^2 + \right.\right. \tag{2.7}$$

$$\left(\frac{s}{h}\right)^3+\left(\frac{s}{h}\right)^4\Big]-\frac{a^2}{e^2\beta}\sum_{i=1}^{2}\Psi_i[A\sigma_{3-i}-B(\sigma_{3-i}-e^2)]+$$

$$\frac{2a_1a^3(2-e^2)}{3h^3K(e)}D\left(1+2\frac{s}{h}\right)-\frac{\Phi}{h^3}\left(1+\frac{s}{h}\right)+O\left(\frac{1}{\lambda^5}\right)\Big\};$$

$$\Psi_i=K_i\left(\frac{x^2}{a^2\sigma_i}+\frac{y^2}{a^2(\sigma_i-e^2)}-1\right)$$

$$\Phi=\frac{2a_1a^5}{15\beta e^2K(e)}\sum_{i=1}^{2}K_i[A\sigma_{3-i}-B(\sigma_{3-i}-e^2)](\sigma_i+\sigma_ie^2-2e^2)$$

$$K_i=\frac{(-1)^i\sigma_i^2(\sigma_i-e^2)^2}{(\sigma_i-1)[E(e)-(1-\sigma_i)K(e)]},\quad D=\frac{\delta-{}^1/_3(Aa^2+Bb^2)}{K(e)}$$

$$s=\frac{a_0a}{K(e)},\quad \sigma_1=\frac{1+e^2}{3}+\beta,\quad \sigma_2=\frac{1+e^2}{3}-\beta$$

$$\beta=\frac{1}{3}(1-e^2+e^4)^{1/2},\quad e=\left(1-\frac{b^2}{a^2}\right)^{1/2}$$

Here e is the eccentricity of the elliptical domain Ω, and $K(e)$ and $E(e)$ are complete elliptic integrals of the first and second kinds.

According to the analysis carried out above, a solution of the problem that is bounded on the contour of the elliptical contact domain can be obtained from (2.7) to terms $O(\lambda^{-5})$. In fact, we require the expression (2.7) for $q(x,y)$ to have the form

$$q(x,y)=C\frac{\theta}{b}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2} \tag{2.8}$$

Multiplying (2.7) and (2.8) by $(1-x^2/a^2-y^2/b^2)^{1/2}$ and then equating coefficients of identical powers of x and y, we obtain a system of three equations, by solving which, we can express the quantities a, e and C in terms of δ, A and B. We then determine the quantity from the first equilibrium condition for the stamp (1.2).

Because it is difficult to solve the system mentioned in the general case, we limit ourselves therein to terms $O(\lambda^{-2})$. The system is then simplified radically and takes the form

$$D\left(1+\frac{s}{h}+\frac{s^2}{h^2}\right)+\frac{a^2}{e^2\beta}\sum_{i=1}^{2}K_i[A\sigma_{3-i}-B(\sigma_{3-i}-e^2)]=C \tag{2.9}$$

$$\frac{a^2}{e^2\beta}\sum_{i=1}^{2}K_i[A\sigma_{3-i}-B(\sigma_{3-i}-e^2)]\sigma_i^{-1}=C$$

$$\frac{a^2}{e^2\beta}\sum_{i=1}^{2}K_i[A\sigma_{3-i}-B(\sigma_{3-i}-e^2)](\sigma_i-e^2)^{-1}=C(1-e^2)^{-1}$$

We first obtain expressions for A and B in terms of C from the last two equations in (2.9). Substituting these results into the first equation in (2.9) and solving, we obtain C. Finally, the solution of system (2.9) can be written in the form

$$A=\frac{K(e)-E(e)}{2a^2e^2}C,\quad B=\frac{E(e)-(1-e^2)K(e)}{2b^2e^2}C \tag{2.10}$$

$$C=\frac{2\delta}{K(e)}\left[1+\frac{2s}{3h}+\left(\frac{2s}{3h}\right)^2+O\left(\frac{1}{\lambda^3}\right)\right]$$

We finally find from (2.8) and (1.2)

$$P=\frac{4\pi a\theta\delta}{3K(e)}\left[1+\frac{2s}{3h}+\left(\frac{2s}{3h}\right)^2+O\left(\frac{1}{\lambda^3}\right)\right] \tag{2.11}$$

Relations (2.10) and (2.11) of determining a, e and δ as a function of the magnitude of the force P by considering A and B as given, or determining A, B and δ, assuming a and e to be given.

3. Solution for a small relative thickness of the layer. To characterize a layer of small relative thickness in the case of a convex domain, we introduce the following dimensionless geometric parameter $\mu=h/\rho_{min}$, where ρ_{min} is the minimum radius of curvature of the contour L of the domain Ω.

The internal (penetrating) asymptotic solution of the integral equation (1.1) has the following form for small values of μ (/1/, Sect.55)

$$q(x,y)\sim\theta(hA)^{-1}\sum_{i=0}^{\infty}(-1)^iD_ih^{2i}\Delta^if(x,y) \tag{3.1}$$

Here Δ is the Laplace operator and the constants A and D_i are determined from the following expansion

$$\frac{u(\operatorname{sh} 2u+2u)}{\operatorname{ch} 2u-1}=\frac{1}{A}\sum_{i=0}^{\infty} D_i u^{2i} \tag{3.2}$$

It can be found that

$$A={}^1/_2,\ D_0=1,\ D_1=0,\ D_2={}^1/_{45} \tag{3.3}$$

The radius of convergence of the series (3.2) is determined by the magnitude of the first non-zero root of the function cosh $2u-1$ in the complex plane u, in this case the convergence will hold for $|u|<\pi$. It hence follows, for instance, that for a stamp of elliptical planform with the semiaxes a and b, for

$$f(x,y)=\exp\left[i\left(p_1\frac{x}{a}+p_2\frac{y}{b}\right)\right] \tag{3.4}$$

formula (3.1) can be used when the following inequality is satisfied:

$$\lambda\mu\,[(1-e^2)\,p_1^2+p_2^2]<\pi \tag{3.5}$$

For small μ the internal solution is applicable in the whole domain Ω with the exception of a narrow annular zone adjacent to the contour L.

A boundary- layer type solution holds in this narrow zone /1/: the relative thickness of the boundary layer is of the order of μ^{-1}. A boundary-layer type solution tends exponentially (as $\exp(-n\mu^{-1})$, where n is the shortest distance from the point $Q\in\Omega$ to the point $P\in L$ referred to $\rho_{\min}$), to the internal solution with distance from the contour L. We furthermore assume that the parameter μ is so small that a solution of boundary-layer type cannot be taken in the computation.

In the case of the polynomial function $f(x,y)$, the series (3.1) that yields the internal solution is truncated. For instance, for a stamp with polynomial base of the form (2.2), the internal solution (3.1) will also be a polynomial of degree N.

It can be shown that for $q(x,y)$ of the form (3.1), condition (1.4) results directly as necessary from the condition for a minimum of the functional (1.3). therefore, it is here necessary to represent the function $q(x,y)$ for a stamp with base (2.2) in the form

$$q(x,y)=\sum_{m,n=0}^{m+n\leqslant N-2} a_{mn}x^m y^n\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \tag{3.6}$$

when considering the problem assuming the variability of the contact domain in the case of the elliptical domain Ω (semiaxes a and b).

Let us now substitute the function $f(x,y)$ of the form (1.1), (2.2) into (3.1), let us perform all the differentiation operations and equate the result obtained to (3.6). We obtain a relation connecting two polynomials of degree N, whose coefficients depend on a_{mn} and b_{kl}. Equating coefficients of identical powers of x and y, we obtain a system of ${}^1/_2\,(N+1)\,(N+2)$ independent algebraic equations. These equations contain ${}^1/_2\,(N-1)\,N$ unknown coefficients a_{mn}, consequently, only ${}^1/_2\,(N-1)\,N$ out of the whole set of coefficients b_{kl} can be arbitrary. The remaining $2N+1$ coefficients b_{kl} must, as in Sect.2, be considered additional unknowns for given values of δ,α,β,a and b.

We will examine the special case when the shape of the stamp base is a paraboloid, i.e., $g(x,y)=Ax^2+By^2\ (A>0,B>0)$, while the force P is applied to the stamp centre of symmetry. On the basis of (3.1), for this case we find the internal asymptotic solution of the problem for small μ that vanishes on the contour L of the elliptical contact domain. According to the gneral scheme elucidated above, we have

$$q(x,y)=\frac{2\theta\delta}{h}\left[\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+O\left(e^{-\frac{n}{\mu}}\right)\right] \tag{3.7}$$

$$A=\frac{\delta}{a^2},\quad B=\frac{\delta}{b^2},\quad P=\frac{\pi ab\theta\delta}{h}$$

It follows from an asymptotic estimate in the first formula of (3.7) that its error will not exceed 5% if $n\geqslant 3\mu$, i.e., if the point $Q\in\Omega$ is removed by more than $3h$ from the contour L along the normal.

Thus, in the case of a variable contact domain for a parabolic stamp interacting with a layer, the contact domain Ω turns out to be elliptic for sufficiently large λ or sufficiently small μ.

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CONTACT PROBLEMS OF THE THEORY OF PLASTICITY FOR COMPLEX LOADING*

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The problem of the interaction between a stamp and an elastic-plastic body for a non-proportional change in the given loads and taking the contact area indeterminacy into account, is considered. It is assumed that the material properties are described by differential-linear or differential-non-linear relationships between the stress rates and the strain rates encompassing a fairly broad group of the theory of plasticity. It is shown that the initial problem in a generalized formulation is equivalent to a certain quasivariational inequality in the displacement velocities. By using a formulation in the form of a quasivariational inequality, existence conditions are studied for the solution, and a numerical method of investigation is proposed and verified.

Note that the method of integral equations, utilized extensively in the theory of elasticity, can be applied successfully only to certain special classes of contact problems of the theory of plasticity /1, 2/. Research on general questions of the theory of contact problems for elastic-plastic bodies refers to individual plasticity models /3—6/, and the numerical methods that have received extensive development are applied to contact problems under complex loading by using heuristic algorithms /7—9/ that require additional investigation and verification.

1. General formulation of the problem. We consider the quasistatic deformation of an elasto-plastic body occupying a domain Ω of a three-dimensional Cartesian space bounded by a piecewise-smooth surface Γ. The displacements and deformations are assumed to be small. We let t denote a monotonically increasing parameter associated with the loading process, which we shall call time. The solution of the problem is considered in at time interval $[0, T]$. We let $u_i(x, t)$, $\varepsilon_{ij}(x, t)$, and $\sigma_{ij}(x, t)$ denote the components of the displacement vector, and of the strain and stress tensors at the point $x = (x_1, x_2, x_3) \in \Omega$ at the time $t \in [0, T]$. We assume the body is in the unstressed and unstrained state at the initial time $t = 0$. We denote differentiation with respect to time by a point, and with respect to the space variables by a comma. The rule of summation over repeated subscripts is used.

It is assumed that the behaviour of the body material under complex laoding can be described by differential linear or differential non-linear relationships of the form

$$\sigma_{ij}^{\cdot} = A_{ijpq}(x, \varkappa_1, \varkappa_2, \ldots, \varkappa_r, \varepsilon_{\xi\eta}^{\cdot})\, \varepsilon_{pq}^{\cdot} \tag{1.1}$$

The function A_{ijpq} is homogeneous of zeroth degree in $\varepsilon_{\xi\eta}^{\cdot}$ or generally independent of $\varepsilon_{\xi\eta}$ in the case of differential linear relationships. We take $\varkappa_1, \varkappa_2, \ldots, \varkappa_r$ to be values of certain functionals of the strain history. Relations for different versions of flow theory and for theories based on the slip concept /10/ can be represented in the form (1.1). Relationships (1.1) are a special version of the theory of elasto-plastic processes /11/. When $A_{ijpq} = A_{ijpq}(x)$ (1.1) correspond to linear elasticity theory for an inhomogeneous anisotropic body. Note that relationships of the form (1.1) can be used for both active loading and unloading processes.

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